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A FROBENIUS CHARACTERIZATION OF RATIONAL SINGULARITY IN 2-DIMENSIONAL GRADED RINGS

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ABSTRACT. A ring R is said to be F-rational if, for every prime P in R, the local ring R_P has the property that every system of parameters ideal is tightly closed (as defined by Hochster-Huneke). It is proved that if R is a 2-dimensional graded ring with an isolated singularity at the irrelevant maximal ideal m, then the following are equivalent:

(1) R has a rational singularity at m.
(2) R is F-rational.
(3) $a(R) < 0$.

Here $a(R)$ (as defined by Goto-Watanabe) denotes the least nonvanishing graded piece of the local cohomology module $H_m(R)$.

The proof of this result relies heavily on the properties of derivations of R, and suggests further questions in that direction; paradigmatically, if one knows that $D(a)$ satisfies a certain property for every derivation D, what can one conclude about the original ring element a?

M. Hochster and C. Huneke have recently introduced the notion of the tight closure of an ideal. They proved, under mild conditions, that a ring with an isolated singularity in which every ideal generated by a system of parameters is tightly closed must be a rational singularity in characteristic 0 [HH, (4.1), (4.2)]. Works by R. Fedder, V. Srinivas, and K. Watanabe have explored the question of whether the converse is true. Fedder has given an affirmative answer in the case of positively graded complete intersection rings having an isolated singularity at the irrelevant maximum ideal. Srinivas and Watanabe have, independently, characterized the property of F-purity in dimension 2, thereby constructing counterexamples to the stronger question posed by Hochster-Huneke: Does a ring with an isolated rational singularity have the property that each of its ideals is tightly closed? (I.e. is such a ring necessarily F-regular?)

The examples of Watanabe and Srinivas show that there exist graded 2-dimensional rational singularity rings which are not even F-pure, much less F-regular. However, in this paper, I give an affirmative answer in dimension 2 to a weaker (but best possible) conjecture: If R is a graded, normal ring with an isolated rational singularity, then R has the property that each of its system of parameter ideals is tightly closed. This property is called F-rational. The conjecture remains open in dimension higher than 2 for rings with an isolated singularity which are not complete intersections.

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The proof depends fundamentally on the use of derivations in characteristic p, a technique which raises some interesting further questions (see [FHH]). In general, if (R, m) is a local or graded ring with coefficient field K, we would like to know that properties of f can be derived from specific knowledge about df where d is the map from R to $\widetilde{\Omega}^1_{R/K}$, the universally finite derivations of R over K [KD, §11]. The technical observations in this paper are essentially equivalent to observing that if I is a system of parameters ideal in a positively graded ring (R, m) such that I is a reduction of the maximal ideal m, then there exists an integer k, depending only on R and I, such that $d(f) \in I^n\widetilde{\Omega}^1_{R/K}$ implies that $f+g \in I^{n-k}$ for some g such that $d(g) = 0$. We further observe that if (R, m) is normal and K is perfect, then $d(g) = 0$ implies that $g \in R^p$ where $R^p = \{r^p | r \in R\}$. The main result, then, could easily have been deduced from these two observations and a theorem of Lipman and Tessier [LT] which states that $\overline{I^2} = I$ if and only if (R, m) has a rational singularity. (Here, $^-$ denotes the integral closure of the ideal.) However, a result of Watanabe [W], and independently of H. Flenner [FL], that $a(R) < 0$ if and only if R has a rational singularity (where $a(R)$ denotes the least nonvanishing graded piece of the local cohomology module $H^{\dim(R)}_m(R)$) makes it possible to simplify somewhat. Thus, in this paper, we will suppress the former point of view (which does not depend on the grading) in favor of the latter; simply noting here that were it possible to prove, in the *nongraded* (2-dimensional) case, that $df \in I^n\widetilde{\Omega}^1_{R/K}$ implies $f+g \in I^{n-k}$ for some g such that $d(g) = 0$ and some k independent of n, then the Lipman-Tessier result would enable us to deduce that F-rationality is equivalent to rational singularity.

1. Notation and definitions

(R, m) will denote either a local ring with maximal ideal m or a positively graded ring with $R_0 = K$ being a field and $m = R_+$. For a ring (R, m) of characteristic p, ${}^1R$ will denote the ring R viewed as an R-module via the Frobenius map, $F: R \to {}^1R$. Similarly, if M is any R-module, then ${}^1M$ will denote $M \otimes_R {}^1R$. Thus, the R-module structure of ${}^1M$ is just $r \cdot m = r^p m$. For an ideal I, $I^{[q]}$ will denote the ideal generated by $\{n^q | n \in I\}$, where $q = p^e$ is a power of the prime characteristic.

Definition 1.1. A Cohen-Macaulay (C-M) ring (R, m) is F-injective if, for some (equivalently, every) system of parameter ideal I, the Frobenius map $R/I \xrightarrow{F} {}^1(R/I)$ is injective.

Remark 1.2. Note that if the dimension of (R, m) is n and $I_d = (x_1^d, \dots, x_n^d)$ is a system of parameters (s.o.p.) ideal, then the local cohomology module $H^n_m(R)$ can be identified with a direct limit of the modules R/I_d independent of the choice of $(x_1, \dots, x_n)$, and a natural Frobenius action is induced on $H^n_m(R)$ by the maps $R/I_d \xrightarrow{F} {}^1(R/I_d)$. It follows that a C-M ring is F-injective if and only if the map induced by the Frobenius on local cohomology is injective. It is also not difficult to deduce, in the C-M case, that it suffices to test just one s.o.p. ideal I, and that this test comes down to the following "contractedness" criterion for F-injectivity which will be used throughout this paper:

If $f^p \in I^{[p]}$, then $f \in I$.

Note further that if (R, m) is a positively graded ring, then we can choose a graded s.o.p. ideal $I_1 = (x_1, \dots, x_n)$ in constructing $H_m^n(R)$, which thereby inherits a natural grading from R that is well known to be independent of the choice of $(x_1, \dots, x_n)$.

Definition 1.3. Let (R, m) be a positively graded ring of dimension n. We will denote by $a(R)$ the least nonvanishing piece of the local cohomology module $H_m^n(R)$, where $n = m(R)$. That is, $a(R) = \sup\{d | [H_m^n(R)]_d \neq 0\}$. A simple application of the DCC property for local cohomology modules shows that $a(R)$ is necessarily finite.

Remark 1.4. Let (R, m) be a positively graded, C-M ring. Let $(y_1, \dots, y_n)$ be a graded s.o.p. Then $a(R)$ can be computed from the Koszul complex $0 \to K^*(y_1, \dots, y_n) \to R/(y_1, \dots, y_n) \to 0$ as follows: For each map (which is compounded out of multiplications by some y_i on each summand), one assigns the appropriate grading on the corresponding copy of R so that the map has degree 0. At the last stage, one gets $[R/(y_1, \dots, y_n)](t)$ as a graded module, where $t = \sum_{i=1}^n \deg(y_i)$. Thus, as a graded module, $H_m^n(R) = \lim[R/(y_1^d, \dots, y_n^d)](dt)$. (Of course, $M(r)$ is the twisted graded module satisfying $[M(r)]_n = [M]_{r+n}$.) The condition that $a(R) < 0$ therefore implies that, for any $f \in R$, either:

(1) $\deg(f) < t$, or

(2) the equivalence class $[f]$ in $H_m^n(R)$ is 0.

But, since, in a C-M ring, the direct limit for $H_m^n(R)$ is exact, we can replace (2) by

(2′) $f \in (y_1, \dots, y_n)$.

Definition 1.5. Let k be a field of characteristic 0, let Y be a normal variety over K, and let $f\colon Y' \to Y$ be a resolution of the singularities of Y. A point $y \in Y$ is called a *rational singularity* if all the higher order direct image sheaves vanish at y; that is, $R^q f_*(\mathscr{O}_{Y'})_y = 0$ for all $q > 0$. This definition is well known to be independent of the resolution f and to reflect certain properties of the local ring $\mathscr{O}_{Y,y}$. In particular, if y is an isolated singularity, then $R^q f_*(\mathscr{O}_{Y'})_y \simeq H_y^{q+1}(\mathscr{O}_y)$, the local cohomology, for $0 < q < \dim(\mathscr{O}_{Y,y}) - 1$, from which it follows that $\mathscr{O}_{Y,y}$ is C-M.

Theorem 1.6. *Let (R, m) be a positively graded, C-M, normal ring, with $R_0 = K$ being a field of characteristic 0, such that R_Q has a rational singularity for every $Q \neq m$ in* Spec(R). *Then R has a rational singularity if and only if $a(R) < 0$.*

Proof. See [W1].

Definition 1.7. Let $I \subset R$, where R has characteristic p. We say that $x \in I^*$, the *tight closure* of I, if there exists $c \in R^0$ such that $cx^q \in I^{[q]}$ for all sufficiently large q of the form $q = p^e$. (Here, R^0 denotes the complement in R of the union of minimal primes.) If $I = I^*$, we say that I is *tightly closed.*

Definition 1.8. R is *weakly F-regular* if every ideal in R is tightly closed. R is *F-regular* if R_P is weakly F-regular for every $P \in \mathrm{Spec}(R)$.

Definition 1.9. R is *F-rational* if every system of parameters ideal in R is tightly closed.

Remark 1.10. The notions of F-rational and F-regular coincide in the case of Gorenstein rings [HH2]. In general, it suffices to test just one system of parameters ideal to determine whether a C-M local ring is F-rational.

Definition/Remark 1.11. We shall describe how to define the notions of F-injective *type*, F-pure *type*, F-regular *type*, and F-rational *type* in characteristic 0, by reduction to characteristic p. For the purposes of this paper, it suffices to think of R as being affine; that is, R has the form $K[X_1, \dots, X_d]/(f_1, \dots, f_t)$ where K is a field, the X_i's are indeterminates, and the f_i's are polynomials.

Given a property W, defined for rings of characteristic $p > 0$, we define a notion of W-type as follows: $R = K[X_1, \dots, X_n]/(f_1, \dots, f_t)$ has *W-type* if there exists a subring $A \subset K$ of mixed characteristic which contains all the coefficients of each f_i and a dense open set $U \subset \max\operatorname{Spec}(A)$ such that, for every $m \in U$, $\frac{A}{m}[X_1, \dots, X_d]/(f_1, \dots, f_t)$ has property W. (See §4 of [HR] for a more general formulation of this reduction.)

Definition 1.12. Let (R, m) be a ring of characteristic p and let S denote the socle of $H^d_m(R)$, where d is the dimension of R. Then R is said to be *F-unstable* if $F(S) \cap S = 0$, where F denotes the natural action induced on $H^n_m(R)$ (see Remark 1.2) by the Frobenius map.

Theorem 1.13. *Let (R, m) be a C-M ring of characteristic p with an isolated singularity at m. Then R is F-rational if and only if R is F-injective and F-unstable.*

Proof. See [FW, Theorem 2.8].

Remark 1.14. In the case that (R, m) is a graded F-injective ring with an isolated singularity, R is F-unstable if and only if $a(R) < 0$. (See [FW, Remark 1.17].)

Definition 1.15. Let R be a ring essentially of finite type over a perfect field K of characteristic $p > 0$. Let T denote the subring $\{r \in R | D(r) = 0$ for every $D \in \operatorname{der}_K(R)\}$. Then we say R/K satisfies D_0 if $T = R^p$.

Definition 1.16. Let R be a ring of characteristic $p > 0$. R is *1-contracted* if:
(a) There exist a non-zero-divisor (NZD) y in R such that y is not a unit.
(b) For every NZD y in R, the condition $f^p \in (y^p)$ implies $f \in (y)$.

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Our objective is to prove that for a positively graded, two-dimensional, normal ring R, $a(R) < 0$ implies R has F-rational type. From the preceding discussion of results already known [HH4 and W1], the proof will actually establish that such an R has a rational singularity if and only if it has F-rational type. Moreover, by Theorem 1.12 and Remark 1.13, it will only be necessary to show that for such R, $a(R) < 0$ implies F-injective type. We begin by making some elementary observations about the property D_0.

Lemma 2.1. *Let $K \subset L$ be an inclusion of fields of characteristic $p > 0$. If K is perfect, then L/K satisfies D_0.*

Proof. There exists a p-basis $S = \{x_\alpha\}$ for L over L^p (i.e. L has a vector space basis over L^p given by all distinct monomials $x_{\alpha_1}^{r_i} \cdots x_{\alpha_n}^{r_n}$ such that

$0 \le r_i \le p-1$). Moreover the vector space of Kähler differentials Ω_{L/L^p} is free over L^p with basis $\{dx_\alpha\}$ (see [M, 26.F]). Hence, there exist (necessarily independent) derivations $\{D_\alpha\} \subset \mathrm{der}_{L^p}(L)$, defined by $D_\alpha(x_\beta) = \delta_\alpha^\beta$. It follows immediately that L/L^p satisfies D_0. But, $K = K^p$ is a subring of L^p, so $\mathrm{der}_{L^p}(L) \subset \mathrm{der}_K(L)$. Hence L/K satisfies D_0. □

By considering the case where L is a separable algebraic extension of K in Lemma 2.1 (whence, L/K satisfies D_0, but $\mathrm{der}_K(L) = 0$, so $L^p = L$), one can immediately derive the standard fact (see [ZS]) which we state below for future reference.

Corollary 2.2. *Let K be a perfect field of characteristic $p > 0$, and let K' be a separable algebraic extension of K. Then, K' is perfect.*

Proposition 2.3. *Let R be a ring which is finitely generated over a perfect field K of characteristic $p > 0$. Let T denote the total quotient ring of R (i.e. $T = S^{-1}R$ where S is the set of non-zero-divisors (NZD) of R). If $R \ne T$, then the following are equivalent:*

(a) *R is 1-contracted.*

(b) *R is reduced and $T^p \cap R = R^p$.*

(c) *R is reduced and R/K satisfies D_0.*

Proof. (a) ⇒ (b). We first show that 1-contractedness implies reduced. If not, then there exists $0 \ne f \in R$ such that $f^p = 0$. Since $R \ne T$, there exists $y \in \mathrm{NZD}(R)$ such that y is not a unit. For each positive integer d, $f^p \in (y^d)^p$. Thus, by 1-contractedness, $f \in y^d R$ for each d. But $\bigcap_{d=1}^\infty y^d R = 0$; so $f = 0$.

Now we show that 1-contractedness implies $T^p \cap R = R^p$. Let $a \in T^p \cap R$. Then $a = (f/y)^p$ for some $f \in R$ and $y \in \mathrm{NZD}(R)$. Thus, we have an equation $f^p = ay^p$ in R. By 1-contractedness, $f = by$ for some $b \in R$. Hence, $f^p = b^p y^p = ay^p$. Since y is an NZD, $b^p = a$. That is, $a \in R^p$ as desired.

(b) ⇒ (c) Let $a \in R$ satisfy $D(a) = 0$ for all $D \in \mathrm{der}_K(R)$. We must show that $a \in R^p$. Let $\{Q_i\}_{i=1}^n$ be the set of minimal primes of R. Then $R_{Q_i} = L_i$ is a field. Denoting by W the multiplicative set $R\backslash Q_i$, we have $\mathrm{der}_K(L_i)$ "="$W^{-1}(\mathrm{der}_K(R))$ because R is finitely generated over K). Let ϑ_i denote the (not necessarily injective) homomorphism from R into R_{Q_i} which sends r to "$r/1$". Then, since $D(a) = 0$ for all $D \in \mathrm{der}_K(R)$, it follows that $D(\vartheta_i(a)) = 0$ for all $D \in \mathrm{der}_K(L_i)$. By Lemma 2.1, L_i/K satisfies D_0. Hence, $\vartheta_i(a) \in L_i^p$ for each $1 \le i \le n$. Since R is reduced, T is 0-dimensional and reduced; so $T \simeq \prod_{i=1}^n L_i$ via the isomorphism $R \to (\vartheta_1(r), \dots, \vartheta_n(r))$. Denote $\vartheta_i(a) = b_i^p$ where $b_i \in L_i$. There exists $b \in T$ such that $\vartheta_i(b) = b_i$ for each $1 \le i \le n$ (where, by abuse of notation, ϑ_i here is the map from T to T_{Q_i} rather than R to R_{Q_i}). It follows that $\vartheta_i(b^p) = b_i^p = \vartheta_i(a)$ for each $1 \le i \le n$. Hence, $a = b^p \in T^p \cap R = R^p$ as claimed.

(c) ⇒ (a). Assume that we have an equation $f^p = ay^p$ where $y \in \mathrm{NZD}(R)$. Let $D \in \mathrm{der}_K(R)$ be arbitrary. Then differentiating the equation, we get $0 = y^p D(a)$. But since $y^p \in \mathrm{NZD}(R)$, $D(a) = 0$. Since R satisfies D_0, $a \in R^p$. Let $a = b^p$ with $b \in R$. Then $f^p = b^p y^p$. But, since R is reduced, $f = by$ as claimed. □

Proposition 2.4. *Let R be a C-M ring finitely generated over a perfect field K of characteristic $p > 0$. Assume further that $\dim(R) \ge 1$. If R_Q is F-injective* *for each Q of height 1 in R, then R is reduced, and R/K satisfies D_0. In particular, if R is normal, then R/K satisfies D_0.*

Proof. Note that, under the hypotheses, R cannot be equal to its total quotient ring. Let $y \in \mathrm{NZD}(R)$ be such that y is not a unit in R. Let $(y) = \bigcap_{i=1}^n Q_i$ be the primary decomposition of (y), and let P_i be the prime ideal associated to Q_i. Since R is C-M, P_i is a height 1 prime for each $1 \leq i \leq n$, and $yR_{P_i} \cap R = Q_i$. Given an equation of the form $f^p = ay^p$, we can pass to R_{P_i} to deduce, by the locally F-injective hypothesis, that $f \in yR_{P_i} \cap R = Q_i$. Hence, $f \in \bigcap_{i=1}^n Q_i = (y)$; and so, R is 1-contracted. By Proposition 2.3, then, R is reduced and R/K satisfies D_0. □

Remark 2.5. The C-M hypothesis in Proposition 2.4 was unnecessary—the proof only requires that principal ideals be unmixed.

In dimension 1, F-injectivity is the same as 1-contractedness. Thus, Proposition 2.3 already gives an interesting characterization of F-injectivity in dimension 1. However, in the graded case, we can push this observation further.

Remark 2.6. Let R be a 1-dimensional positively graded, reduced, C-M ring finitely generated over a perfect field K of characteristic $p > 0$. Then R is F-rational if and only if $a(R) < 0$ if and only if R is regular. That F-rational implies $a(R) < 0$ follows from [FW, Remark 1.17 and Proposition 2.4]. To prove $a(R) < 0$ implies R is regular, under the hypotheses, we can easily reduce to the domain case. Then, identifying $H_m^1(R)$ with $[R/(y)](t)$ (where $t = \deg(y)$ is the least positive degree in R) it follows that for $z \in m$, $\deg(z) \geq \deg(y)$, and therefore by Remark 1.4, $z \in (y)$. Hence, m is generated by y, and R is regular. □

We now turn our attention to the more difficult 2-dimensional case.

Definition 2.7. Let R be a ring and K a field contained in R. We say that a subset $S \subset \mathrm{der}_K(R)$ is D-complete if $\{a \in R | D(a) = 0 \text{ for all } D \in \mathrm{der}_K(R)\} = \{a \in R | D(a) = 0 \text{ for all } D \in S\}$. Note that any set of generators for $\mathrm{der}_K R$ is a D-complete set. However, a D-complete set need not generate $\mathrm{der}_K(R)$.

Definition 2.8. Let R be a positively graded subring over a field $K = [R]_0$, and let S be a D-complete set of homogeneous derivations. Then $d_S(R)$ will denote $\sup\{\deg(D) | D \in S\}$.

Example 2.9. Let $R = K[X_0, \dots, X_n]/(G)$ be an irreducible, homogeneous, normal, hypersurface ring of dimension n. Define

$$D_{ij} = G_{x_i}\frac{\partial}{\partial x_j} - G_{x_j}\frac{\partial}{\partial x_i}.$$

A priori, $D_{ij} \in \mathrm{der}_K(K[X_0, \dots, X_n])$. However, since $D_{ij}(G) = 0$, it follows that $D_{ij} \in \mathrm{der}_K(R)$ as well. Let L denote the fraction field of R. Then, L has transcendency degree n over K. Assuming that at least one of the partial derivatives (say G_{x_0}) is nonzero, then the Jacobian matrix T has rank 1. It follows (see [M, 27.B], or see later in this paper for further details on this point) that $\Omega_{L/K}$ and $\mathrm{der}_K(L)$ are vector spaces of dimension n. It is also easy to verify that the derivations $D_{0j} \in \mathrm{der}_K(R) \hookrightarrow \mathrm{der}_K(L)$ are linearly independent for $1 \le j \le n$ (if $D = \sum_{i=1}^{n} l_i D_{0i} = 0$, just compute $D(x_i)$ to see that $l_i = 0$). If $D \in \mathrm{der}_K(R)$ is arbitrary, $D = \sum_{j=1}^{n} l_j D_{0j}$ for some $l_j \in L$. Clearing denominators, there exists $s \in R$ such that sD lies in the R-span of $\{D_{0j}\}_{j=1}^{n}$. Since $s \in \mathrm{NZD}(R)$, $D(a) = 0$ if and only if $sD(a) = 0$. It follows that $S = \{D_{0j}\}$ is a D-complete set in $\mathrm{der}_K(R)$. More specifically, let F be $K[X, Y, Z]/(X^2+Y^3+Z^5)$ with $\deg(X) = 15$, $\deg(Y) = 10$, and $\deg(Z) = 6$. Then, $D_1 = 3y^2\partial/\partial x - 2x\partial/\partial y$ and $D_2 = 5z^4\partial/\partial x - 2x\partial/\partial z$ form the set $S = \{D_1, D_2\}$ described above (assuming characteristic $\neq 2$). Since $\partial/\partial x$ has degree -15 and y^2 has degree 20, D_1 has degree 5. Similarly D_2 has degree 9. Therefore, $d_S(R) = 9$. However, we can make a better choice by replacing D_2 with $D_3 = 15x\partial/\partial x + 10y\partial/\partial y + 6z\partial/\partial z$ which has degree 0. Then $T = \{D_1, D_3\}$ is clearly a linearly independent set (assuming characteristic $\neq 2$ or 3). But $d_T(R) = 5$. Given our next result (Theorem 2.10) this calculation shows that R is F-rational whenever the characteristic of K is bigger than 5. Because R is a complete intersection, the result of [F2] had previously shown that R is F-rational, provided that the characteristic of K is bigger than 31.

Theorem 2.10. *Let (R, m) be a positively graded Cohen-Macaulay ring of dimension 2 such that $R_0 = K$ is a perfect field of characteristic $p > 0$. Assume further that R_Q is F-injective for each homogeneous prime Q of height one. Let S be a D-complete set in R. If $a(R) < 0$ and if $d_S(R) = N < p$, then R is F-rational.*

Proof. Let (y_1, y_2) be a homogeneous system of parameters in m. The condition $a(R) < 0$ means: if x is homogeneous satisfying $\deg(x) \ge \deg(y_1) + \deg(y_2)$, then $x \in (y_1, y_2)R$. (See Remark 1.4.) Given a homogeneous equation $f^p = a_1 y_1^p + a_2 y_2^p$, we get, by differentiating, $D(a_1)y_1^p \in (y_2^p)$ for every $D \in S$. Since $y_1^p \in \mathrm{NZD}(R/(y_2^p))$, $D(a_1) \in (y_2^p)$. Every homogeneous element except for 0 in the ideal (y_2^p) has degree greater than or equal to $p\deg(y_2)$. thus, there are 2 cases:

(1) $D(a) = 0$ for all $D \in S$ or

(2) there exists $D \in S$ such that $\deg(D(a_1)) \ge p\deg(y_2)$.

In the first case it follows that $D(a) = 0$ for all $D \in \mathrm{der}_K(R)$. By Proposition 2.4, then R is reduced and satisfies D_0. Thus, $a_1 = b_1^p \in R^p$, and $(f - b_1 y_1)^p = a_2 y_2^p$. By the same reasoning, $a_2 = b_2^p \in R^p$ and $f = b_1 y_1 + b_2 y_2$ (because R is reduced).

In the second case, $N + \deg(a_1) \ge \deg(D(a_1)) \ge p\deg(y_2)$. But, since $f^p = a_1 y_1^p + a_2 y_2^p$, $p\deg(f) \ge \deg(a_1) + p\deg(y_1)$. Therefore $p\deg(f) \ge p\deg(y_1) + p\deg(y_2) - N$, and, so, $p(\deg(y_1) + \deg(y_2) - \deg(f)) \le N < p$. It follows that $\deg(f) \ge \deg(y_1) + \deg(y_2)$; whence, since $a(R) < 0$, $f \in (y_1, y_2)$.

We conclude, from the analysis of each case, that R is F-injective. Since R is F-injective and $a(R) < 0$, R is F-rational (see [FW, Remark 1.17 and Theorem 2.8]). □

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Although the characteristic 0 version of Theorem 2.10 is less technical to state, its proof will require a somewhat tedious determination of what information can be preserved in reducing from characteristic 0 to characteristic p.

Let us begin by fixing notation for the remainder of the paper. S will denote the polynomial ring $K[X_1, \dots, X_n]$, where K is a field of characteristic 0. R will denote the quotient of S by a quasi-homogeneous ideal I. Let $A \subseteq K$ be a ring of mixed characteristic such that some fixed set of generators $(G_1, \dots, G_r)$ of I is contained in $A[X_1, \dots, X_n]$. Then, for any maximal ideal $\mu \subseteq A$, we will denote by K_μ, S_μ, and R_μ respectively the field A/μ, the polynomial ring $K_\mu[X_1, \dots, X_n]$, and the quotient ring $S_\mu/(G_1, \dots, G_r)S_\mu$. We will denote (ambiguously) by m the homogeneous maximal ideal of whatever graded ring is in question, be it S, R, S_μ, or R_μ. Capital letters will be used to denote homogeneous polynomials in S or S_μ, and small letters for their images in R or R_μ. The symbol $^-$ will be employed, when it seems helpful, to signify the passage from $A[X_1, \dots, X_n]$ to $K_\mu[X_1, \dots, X_n]$, $\mathbf{Z}$ will denote the integers viewed as a subring of K.

In practice, the subrings of mixed characteristic that we will construct inside K will all be finitely generated over $\mathbf{Z}$ (thus, assuring that they will have mixed characteristic). They will be obtained by adjoining to $\mathbf{Z}$ all the (finitely many) coefficients of all the polynomials from S or R that we need to preserve various finite systems of equations in $A[X_1, \dots, X_n]$ (and, therefore, in S_μ or R_μ). The properties defined by these systems of equations will then be seen to pass from characteristic 0 to characteristic p. Since we will ultimately need to pass several properties at once, we will need to expand A repeatedly. This leads to an exceedingly clumsy statement of the form: "There exists $A_0 \subset K$ such that for every $A \subset K$ finitely generated over A_0, there is a dense open set $U \subset \max\operatorname{Spec}(A)$ such that, for every $\mu \in U, \dots$" we will abbreviate this clumsy formulation with the simpler and more suggestive phrase: "for almost every $\mu, \dots$".

The following elementary lemmas, generalize some simple observations in [F2], where the only concern was the case when $R = S$ is regular.

Lemma 3.1. *Let $\{h, f_1, \dots, f_d\} \subset R$ satisfy $h \in (f_1, \dots, f_d)R$ (respectively, $h \in (f_1, \dots, f_d)R_m$). Then, for almost every μ, $\overline{h} \in (\overline{f}_1, \dots, \overline{f}_d)R_\mu$ (respectively, $\overline{h} \in (\overline{f}_1, \dots, \overline{f}_d)(R_\mu)_m$).*

Proof. See [F2, Lemma 2.4], for the case $R = S$. For $R = S/(G_1, \dots, G_t)$, simply apply the already established result to the set $\{H, F_1, \dots, F_d, G_1, \dots, G_t\} \subset S$. $\square$

Lemma 3.2. *Let R be reduced, and let $0 \neq f \in R$. Then, for almost every μ, $0 \neq \overline{f} \in R_\mu$.*

Proof. Since $R = S/(G_1, \dots, G_t)$, we have $F \notin (G_1, \dots, G_t)S$. Let $\overline{K}$ denote the algebraic closure of K. The radical ideal $(G_1, \dots, G_t)$ has a prime decomposition $(G_1, \dots, G_t) = \bigcap_{i=1}^d P_i$. If S' is any integral extension of S; there exists, for each $1 \leq i \leq d$, $Q_i \in \operatorname{Spec}(S')$ such that $Q_i \cap S = P_i$. It follows that the radical of $(G_1, \dots, G_t)S'$ is contained in $\bigcap_{i=1}^d Q_i$. But, since

$$F \notin (G_1, \dots, G_t)S = \bigcap_{i=1}^{d} P_i = \bigcap_{i=1}^{d}(Q_i \cap S), \qquad F \notin \bigcap_{i=1}^{d} Q_i.$$

Applying the reasoning above to the ring $S' = \overline{K}[X_1, \dots, X_n]$, we find that there exists $\{\alpha_1, \dots, \alpha_n\} \subset \overline{K}$ such that $G_1(\alpha_1, \dots, \alpha_n) = \cdots = G_t(\alpha_1, \dots, \alpha_n)$

$= 0$, but $F(\alpha_1, \dots, \alpha_n) \neq 0$. Construct $A_0 \subset K$ by adjoining to $\mathbf{Z}$ all the coefficients of the polynomials $F, G_1, \dots, G_t$. Construct $A_0' \subset \overline{K}$ by adjoining to A_0 the elements $\alpha_1, \dots, \alpha_n$. Let A be any algebra finitely generated over A_0, and let $A' = A[\alpha_1, \dots, \alpha_n]$. Since $0 \neq F(\alpha_1, \dots, \alpha_n) \in A'$, there exists a proper closed set $X' = V(F(\alpha_1, \dots, \alpha_n)) \subset \operatorname{Spec}(A')$ such that for every maximal ideal μ' in A' with $\mu' \notin X'$, we have $G_1(\overline{\alpha}_1, \dots, \overline{\alpha}_n) = \cdots = G_t(\overline{\alpha}_1, \dots, \overline{\alpha}_n) = 0$ in $K_{\mu'} = A'/\mu'$, but $F(\overline{\alpha}_1, \dots, \overline{\alpha}_n) \neq 0$. Hence, $\overline{F} \notin (\overline{G}_1, \dots, \overline{G}_t)S_{\mu'}$, for almost every μ' (where $S_{\mu'}$ denotes $K_{\mu'}[X_1, \dots, X_n]$). If i denotes the inclusion map from A to A', then the map $i^*\colon \operatorname{Spec}(A') \to \operatorname{Spec}(A)$ is a closed map (by integrality). Therefore, $i^*(X') = X$ is a proper closed set in $\operatorname{Spec}(A)$. Of course, for $\mu' \in \max\operatorname{Spec}(A')$, $i^*(\mu') = \mu' \cap A = \mu$ is a maximal ideal in A (by integrality). Moreover, for every $\mu \notin X$, there exists $\mu' \notin X'$ such that $i^*(\mu') = \mu$. Since, for any such pair μ and μ', $S_\mu = (A/\mu)[X_1, \dots, X_n] \subseteq A'/\mu'[X_1, \dots, X_n] = S_{\mu'}$, the desired conclusion that $\overline{F} \notin (\overline{G}_1, \dots, \overline{G}_t)S_\mu$ follows from the fact that $\overline{F} \notin (\overline{G}_1, \dots, \overline{G}_t)S_{\mu'}$. □

Proposition 3.3. *Let R be a reduced ring. Then*

(1) $\dim((R_\mu)_m) = \dim(R_m)$ *for almost every μ.*

(2) *If R has the property that R_m and $(R_\mu)_m$ are C-M for almost every μ, then* $\operatorname{ht}(JR_m) = \operatorname{ht}(\overline{J}(R_\mu)_m)$ *for each ideal $J \subseteq mR$, for almost every u. In particular, if $J = (f_1, \dots, f_d)$ and $\{y_1, \dots, y_s\}$ is a maximal regular sequence in $(f_1, \dots, f_d)R_m$, then $\{\overline{y}_1, \dots, \overline{y}_s\}$ is a maximal regular sequence in $(\overline{f}_1, \dots, \overline{f}_d)(R_\mu)_m$ for almost every μ.*

Proof. We first prove (2) under the assumption that (1) has already been established for the ring R in question. $R_m = (S/(G_1, \dots, G_t))_m$ has dimension $r \geq s$. There exists $y_{s+1}, \dots, y_r$ such that $y_1, \dots, y_r$ forms an s.o.p. (hence, maximal regular sequence) for m in R_m. Construct A_0 by adjoining to $\mathbf{Z}$ all the coefficients of all the polynomials $G_1, \dots, G_t, F_1, \dots, F_d, Y_1, \dots, Y_r$, plus whatever coefficients are necessary to guarantee both that $\dim((R_\mu)_m) = \dim(R_m)$ and that $(R_\mu)_m$ is C-M for almost every μ. For each $1 \leq i \leq s$, we have equations $y_i \in (f_1, \dots, f_d)$. Repeated application of Lemma 3.1 allows us to construct $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_s = B_0$ such that B_0 is finitely generated over A_0; and, for every A finitely generated over A_i, $\overline{y}_i \in (\overline{f}_1, \dots, \overline{f}_d)R_\mu$ for all μ in some dense open set $U_i \subseteq \max\operatorname{Spec}(A)$. Since $(y_1, \dots, y_r)$ is an s.o.p. ideal, we have equations $x_j^{e_j} \in (y_1, \dots, y_r)R_m$ for each $1 \leq j \leq n$. Again applying Lemma 3.1, we can construct $B_0 \subseteq B_1 \subseteq \cdots \subseteq B_n = C$ such that each B_i is finitely generated over $\mathbf{Z}$ and, for every A finitely generated over B_i, $\overline{x}_j^{e_j} \in (\overline{y}_1, \dots, \overline{y}_r)(R_\mu)_m$ for all μ in some dense open set $W_j \subseteq \max\operatorname{Spec}(A)$.

It follows that if A is any ring finitely generated over C, then both sets of equations, $\overline{y}_i \in (\overline{f}_1, \dots, \overline{f}_d)R_\mu$ for $1 \leq i \leq s$ and $x_j^{e_j} \in (\overline{y}_1, \dots, \overline{y}_r)(R_\mu)_m$ for $1 \leq j \leq n$ hold as long as μ is in the dense open set

$$V = \left(\bigcap_{i=1}^{s} U_i\right) \cap \left(\bigcap_{j=1}^{n} W_j\right) \subseteq \max\operatorname{Spec}(A).$$

Thus, $(\overline{y}_1, \dots, \overline{y}_r)$ is an m-primary ideal in $(R_\mu)_m$. Since $\dim((R_\mu)_m) = r$, and $(R_\mu)_m$ is C-M, $\overline{y}_1, \dots, \overline{y}_r$ must be a regular sequence. Denoting

$(\overline{f}_1, \ldots, \overline{f}_d)$ by $\overline{J}$, it follows that $\overline{y}_1, \ldots, \overline{y}_s$ is a regular sequence in $\overline{J}$, and so $\mathrm{ht}(\overline{J}) = \mathrm{gr}(\overline{J}) \geq \mathrm{gr}(J) = \mathrm{ht}(J)$.

To prove the reverse inequality, suppose by way of contradiction that $\{y_1, \ldots, y_s\}$ is a maximal regular sequence in JR_m, but that $\{\overline{y}_1, \ldots, \overline{y}_s\}$ is a regular sequence in $\overline{J}(R_\mu)_m$ that is not maximal for almost every μ. Then, we can choose $y_{s+1} \in JR$ such that $\overline{y}_{s+1} \in \overline{J}R_\mu$ and $\{\overline{y}_1, \ldots, \overline{y}_{s+1}\}$ forms a regular sequence for almost every μ. Let T denote the ring $R/(y_1, \ldots, y_s) = S/(Y_1, \ldots, Y_s, G_1, \ldots, G_t)$. Since $\{y_1, \ldots, y_s\}$ is a maximal regular sequence in J and $y_{s+1} \in J$, we must have $y_{s+1} \in \mathrm{ZD}(T_m)$. Let $0 \neq f \in T$ be such that $fy_{s+1} = 0$ in T_m. By Lemma 3.2, $\overline{f} \neq 0$ in $(T_\mu)_m$ for almost every μ. But, by Lemma 3.1, $\overline{f}\overline{y}_{s+1} = 0$ in $(T_\mu)_m$ for almost every μ. This contradicts the claim that $\{\overline{y}_1, \ldots, \overline{y}_{s+1}\}$ forms a regular sequence in $(R_\mu)_m$. Thus, $\{\overline{y}_1, \ldots, \overline{y}_s\}$ must have already been a maximal regular sequence in $\overline{J}(R_\mu)_m$, and so $s = \mathrm{ht}(J) = \mathrm{gr}(J) = \mathrm{gr}(\overline{J}) = \mathrm{ht}(\overline{J})$ as desired.

To prove (1), we simply apply the result in (2) to the special case where $R = S$. In this case, both hypotheses are self-evident—that $\dim((S_\mu)_m) = \dim(S_m)$ and that S_m and $(S_\mu)_m$ are C-M for every μ. Since $\dim(R_m) = \dim(S_m) - \mathrm{ht}((G_1, \ldots, G_t)S_m)$ and $\dim((R_\mu)_m) = \dim((S_\mu)_m) - \mathrm{ht}((\overline{G}_1, \ldots, \overline{G}_t)(S_\mu)_m)$, the result is immediate. □

Proposition 3.4. *Let R be a reduced ring such that R_m is C-M. Then:*

(1) *$(R_\mu)_m$ is C-M for almost every μ.*

(2) *If $F_*: 0 \to R_m^{n_d} \xrightarrow{\vartheta_d} R_m^{n_{d-1}} \to \cdots \xrightarrow{\vartheta_2} R_m^{n_1} \xrightarrow{\vartheta_1} R_m^{n_0} \to M \to 0$ is a minimal free resolution of the R_m module M, with ϑ_i being an $n_i \times n_{i-1}$ matrix with entries in R, then $(F_\mu)\colon 0 \to (R_\mu)_d^{n_d} \xrightarrow{\overline{\vartheta_d}} (R_\mu)_m^{n_{d-1}} \to \cdots \xrightarrow{\overline{\vartheta_2}} (R_\mu)_m^{n_1} \xrightarrow{\overline{\vartheta_1}} (R_\mu)_m^{n_0} \to \overline{M} \to 0$ is a minimal free resolution of $\overline{M}$ for almost every μ (here, $\overline{\vartheta}_i$ denotes the matrix obtained by reducing each of the entries of ϑ_i module μ, and $\overline{M}$ denotes the cokernel of $\overline{\vartheta}_1$).*

Proof. We first prove (2) under the assumption that R is a ring for which (1) has already been established. Clearly, since $\vartheta_i\vartheta_{i-1} = 0$, $\overline{\vartheta}_i\overline{\vartheta}_{i-1} = 0$ for almost every u (Lemma 3.1). Therefore, $(F_\mu)_*$ is a complex. By the Buchsbaum-Eisenbud criterion [BE], the complex is exact if and only if

(a) $\mathrm{rk}(\overline{\vartheta}_{i+1}) + \mathrm{rk}(\overline{\vartheta}_i) = n_i$ for each i,

(b) $I(\overline{\vartheta}_i)$ contains a regular sequence of length i.

Here, of course, the rank of each matrix $\overline{\vartheta}_i$ is the size of its largest nonvanishing minor, and the corresponding ideal $I(\overline{\vartheta}_i)$ is the ideal generated by all of the minors whose size is exactly $\mathrm{rk}(\overline{\vartheta}_k)$.

The Buchsbaum-Eisenbud criterion applied to F_* guarantees that (a) and (b) hold if ϑ_i is used in place of $\overline{\vartheta}_i$. By Proposition 3.3(2), therefore, it suffices to show that $\mathrm{rk}(\overline{\vartheta}_i) = \mathrm{rk}(\vartheta_i)$ for each i, for almost every μ. (Note that $I(\vartheta_i)$ and $I(\overline{\vartheta}_i)$ stand in relation to each other as $(f_1, \ldots, f_d)$ and $(\overline{f}_1, \ldots, \overline{f}_d)$ in Proposition 3.3, provided that $\mathrm{rk}(\vartheta_i) = \mathrm{rk}(\overline{\vartheta}_i)$.) If $\mathrm{rk}(\vartheta_i) = k$, then every $(k+1) \times (k+1)$ determinantal minor of ϑ_i is 0 in R_m. Therefore, each $(k+1) \times (k+1)$ minor of $\overline{\vartheta}_i$ is 0 in $(R_\mu)_m$ for almost every μ, and so $\mathrm{rk}(\overline{\vartheta}_i) \leq k$. But since $\mathrm{rk}(\vartheta_i) = k$, there exists a $k \times k$ minor Δ whose determinant is not 0 in R_m. Therefore, by Lemma 3.2, $\overline{\Delta}$ is not 0 in $(R_\mu)_m$ for almost every m, and so $\mathrm{rk}(\overline{\vartheta}_i) = k$. Of course, the minimality of $(F_\mu)_*$ follows immediately from the minimality of F_*—all the entries of ϑ_i lie in m; therefore, so do all the entries of $\overline{\vartheta}_i$.

(1) To apply Proposition 3.3 in the argument above, we needed to assume that $(R_\mu)_m$ is C-M for almost every μ. However, this assumption is gratuitous, as can be seen by applying the argument above to the special case $R = S$. It is self-evident that $(S_\mu)_m$ is regular (hence, C-M) for every μ. Since S_m is regular and local, $R_m = (S/I)_m$ has a finite minimal free resolution as S_m-module. By the result just proved, then; $(R_\mu)_m$ has a minimal free resolution of exactly the same form as $(S_\mu)_m$ module. In particular, $pd_{S_m}(R_m) = pd_{(S_\mu)_m}(R_\mu)_m$. Obviously, $\dim(S_m) = \dim((S_\mu)_m)$. Therefore, by The Auslander-Buchsbaum formula (see [Se, IV, Proposition 21]), $\mathrm{depth}_m(R_\mu)_m = \mathrm{depth}_m(R_m) = \dim R_m$. But, by Proposition 3.3(1), $\dim(R_m) = \dim((R_\mu)_m)$. Thus $(R_\mu)_m$ is C-M for almost every μ. □

Proposition 3.5. *If R is a C-M, reduced, positively graded ring, then $a((R_\mu)) = a(R)$ for almost every μ.*

Proof. Goto and Watanabe [GW] show that $a(R) = \max\{t | [H^d_m(R)]_t \neq 0\} = -\min\{t | [\omega]_t \neq 0\}$ where $d = \dim(R)$ and ω is the canonical module of R. Denote the canonical module of S by Ω, and the canonical modules of S_μ and R_μ by Ω_μ and ω_μ respectively. Let $t = \dim(S) - \dim(R)$. By Proposition 3.3 (since it suffices to check each dimension in question at the irrelevant ideal m), $t = \dim(S_\mu) - \dim(R_\mu)$ for almost every μ. Thus, $\omega = \mathrm{Ext}^t_s(R, \Omega)$ and $\omega_\mu = \mathrm{Ext}^t_{s_\mu}(R_\mu, \Omega_\mu)$ (see [GW, Proposition 2.1.6]). Let $F_*: 0 \to F_k \xrightarrow{\vartheta_{k-1}} F_{k-1} \to \cdots \to F_1 \xrightarrow{\vartheta_1} S \to S/I \to 0$ be a minimal graded free resolution of R. Then, of course, $(F_*)_m$ will be a minimal free resolution of R_m (since all the entries of each ϑ_i lie in the irrelevant ideal). Since

$$pd_{S_m}(R_m) = \dim(S) - \mathrm{depth}_m(R) = \dim(S) - \dim(R),$$

it follows that $k = t$. Moreover,

$$(F_\mu)_*: 0 \to (F_\mu)_t \xrightarrow{\overline{\vartheta_{t-1}}} (F_\mu)_{t-1} \to \cdots \to (F_\mu)_1 \xrightarrow{\overline{\vartheta_1}} S_\mu \to R_\mu \to 0$$

has the property (Proposition 3.4) of being a graded complex such that $(F_{\mu*})_m$ is exact for almost every μ. It follows that $F_{\mu*}$ is exact for almost every μ (the homology is a graded module which vanishes after localizing at the irrelevant ideal). As a graded S-module (respectively S_μ-module), each F_i (respectively $(F_\mu)_i$) is isomorphic to $\bigoplus \sum_{j=1}^{\mathrm{rk}(F_i)} S(l_{ij})$ (respectively $\bigoplus \sum_{j=1}^{\mathrm{rk}((F_i)\mu)} S(l'_{ij})$). The appropriate twists l_{ij} are uniquely determined, inductively, by the degree of each of the entries of ϑ_i (respectively $\overline{\vartheta_i}$), and are chosen so that each map ϑ_i (respectively, $\overline{\vartheta}_i$) will be a homogeneous map of degree 0. Each entry of $\overline{\vartheta}_i$, being obtained from the corresponding entry of ϑ_i by reducing modulo μ, is either 0 or has the same degree as its corresponding entry in ϑ_i. By Lemma 3.2, however, if the entry in ϑ_i is nonzero, then it will be nonzero for almost every μ. Thus, $l_{ij} = l'_{ij}$; that is, the resolutions of R and R_μ have the "same" grading for almost every μ.

As a graded module, $\Omega \simeq S(-n)$ and $\Omega_\mu \simeq S_\mu(-n)$ (see [GW]). Hence, $\operatorname{Hom}_S(F_*, \Omega)$ (respectively, $\operatorname{Hom}_{S_\mu}(F_{\mu *}, \Omega_\mu)$ gives a graded minimal free resolution of ω (respectively ω_μ). That is,

$$\omega \simeq \operatorname{Hom}_s(F_t, S(-n))/\operatorname{im}(\vartheta_{t-1}^T) \quad \text{and} \quad \omega_\mu \simeq \operatorname{Hom}_{S_\mu}((F_\mu)_t, S_\mu(-n))/\operatorname{im}(\overline{\vartheta}_{t-1}^T)$$

(where A^T denotes the transpose of the matrix A). In particular, since

$$\operatorname{Hom}_S(F_t, S(-n)) \simeq \bigoplus_{j=1}^{r} S(-l_{ij} - n)$$

and

$$\operatorname{Hom}_{S_\mu}((F_\mu)_t, S_\mu(-n)) \simeq \bigoplus_{j=1}^{r} S(-l_{ij} - n), \quad \text{where } r = \operatorname{rk}(F_t);$$

w is generated by homogeneous elements $\{e_1, \dots, e_r\}$ such that $\deg(e_j) = (l_{ij}+n)$ and w_μ is generated by homogeneous elements $\{\overline{e}_1, \dots, \overline{e}_r\}$ such that $\deg(\overline{e}_j) = (l_{ij} + n)$. It follows that $a(R) = -\min\{\deg(e_j)\} = -\min\{\deg(\overline{e}_j)\} = a(R_\mu)$ for almost every μ. $\square$

Theorem 3.6. *Let (R, m) be a positively graded C-M normal ring of dimension 2 such that $[R]_0 = K$ is a field of characteristic 0. Then, the following are equivalent*:

(1) *R is F-rational (type).*
(2) *R has a rational singularity.*
(3) *$a(R) < 0$.*

Proof. The fact that F-rational implies rational singularity is a result due to Hochster and Huneke [HH2]. The fact that rational singularity is equivalent to $a(R) < 0$ is a result due to Watanabe and quoted here as Theorem 1.6 Thus, we need only be able to deduce (3) $\Rightarrow$ (1), by invoking Theorem 2.10. However, the application of 2.10 depends on several properties "carrying over" from characteristic 0 to characteristic p.

Since R is positively graded, R has the form S/I where $S = K[X_1, \dots, X_n]$ and $I = (G_1, \dots, G_d)$ is generated by quasi-homogeneous forms in the X_i (we only require $\deg(X_i) > 0$). By Proposition 3.4, R_u will be C-M for almost every μ. By Corollary 3.5, $a(R_\mu) < 0$ for almost every μ. The normality condition comes down to checking that the ideal defining the singular locus has height (= grade) 2. Of course, this ideal is just the Jacobian of I, computed by taking the maximal minors of the matrix $(\partial G_i/\partial X_j)$ (respectively, $(\partial \overline{G}_i/\partial X_j)$) for $1 \le i \le d$ and $1 \le j \le n$. By Proposition 3.3, the fact that the Jacobian ideal contains a regular sequence of length 2 implies that the Jacobian ideal of R_μ contains a regular sequence of length 2 for almost every μ. Thus, R_μ is normal for almost every μ; and so $(R_\mu)_P$ is regular (hence, F-injective) for every P such that $\operatorname{ht}(P) \le 1$, for almost every μ. It remains only to calculate the number $d_S(R_\mu)$ in such a way that it is independent of μ. We will do this by choosing a fundamental set of *graded* derivations S in $\operatorname{der}_K(R)$ in such a way that the "same" set of derivations S_μ, when viewed in $\operatorname{der}_{K_\mu}(R_\mu)$, form a fundamental set for almost every μ. Let $\Omega_{T/L}$ denote the module of Kähler differentials of a given T over a given L. By the second isomorphism theorem for differentials [Mats], we have exact sequences:

$$(*)\qquad \begin{gathered} I/I^2 \xrightarrow{\delta} \Omega_{S/K}\otimes_S R \to \Omega_{R/K}\to 0 \quad \text{and} \\ \overline{I}/\overline{I}^2 \xrightarrow{\overline{\delta}} \Omega_{S_\mu/K_\mu}\otimes_S R_\mu \to \Omega_{R_\mu/K_\mu}\to 0 \end{gathered}$$

where $\delta(G_i) = d(G_i)\otimes 1$, and likewise for $\overline{\delta}$. (Here $\overline{I}$ denotes $(\overline{G}_1, \dots, \overline{G}_t)S_\mu$.) Identifying $\Omega_{S/K}$ with S^n and Ω_{S_μ/K_μ} with S_μ^n (via $e_i \leftrightarrow dx_i$), and mapping R^d (respectively, R_μ^d) onto the generators G_i (respectively, $\overline{G}_i$) of I/I^2 (respectively, $\overline{I}/\overline{I}^2$), the exact sequences $(*)$ become

$$(**)\qquad R^d \xrightarrow{\delta} R^n \to \Omega_{R/K}\to 0, \quad R_\mu^d \xrightarrow{\overline{\delta}} R_\mu^n \to \Omega_{R_\mu/K_\mu}\to 0$$

where δ is just the Jacobian matrix, whose ijth entry is $\partial G_i/\partial X_j$ (reduced modulo I), and likewise for $\overline{\delta}$. Thus, applying $\mathrm{Hom}_R(\_, R)$ to $(**)$, we realize $\mathrm{der}_K(R)$ (respectively, $\mathrm{der}_{K_\mu}(R_\mu)$) as the kernel of the transposed matrix δ^t (respectively $\overline{\delta}^t$). Note that, from comparing $(*)$ with $(**)$ and identifying $\mathrm{Hom}_R(\Omega_{S/K}\otimes_S R, R)$ with $\mathrm{Hom}_S(\Omega_{S/K}, S)\otimes_S R$, it is apparent that an element $(a_1, \dots, a_n)$ in the kernel of δ^t corresponds to the derivation $\sum_{i=1}^n a_i\partial/\partial X_i$ and likewise for the kernel of $\overline{\delta}^t$.

The ring, R (respectively, R_μ), being normal, is a product of domains; therefore, we can reduce to the case of domains. Let L (respectively, L_μ) be the fraction field of R (respectively, R_μ). Let r denote $\mathrm{rk}(\delta) = \mathrm{rk}(\delta^t)$. As noted previously (in Proposition 3.4), r will also be the rank of the matrix $\overline{\delta}$ for almost every μ. Tensoring with the fraction field will not affect the rank. Thus, there exist $n-r$ vectors, $B_i = (b_{i_1}, \dots, b_{in})$ for $r < i \le n$, such that $\{B_{r+1}, \dots, B_n\}$ form a basis for $\ker(\delta^t)$ in L^n. Since $b_{ij}\in L$, we can clear denominators and assume that $B_i \in R^n$. The ith column vector of δ^t is $(\partial G_i/\partial X_1, \dots, \partial G_i/\partial X_n)$. Therefore, if we give R^n the twisted grading $\bigoplus\sum_{i=1}^n R(d_i)$, where $d_i = \deg(X_i)$, and if we give R^d the twisted grading $\bigoplus\sum R(e_i)$, where $e_i = \deg G_i$ the map $R^n \xrightarrow{\delta^t} R^d$ becomes a graded map; so we can assume that the b_{ij}'s are all homogeneous and satisfy $\deg(b_{ij}) - d_j = \deg(b_{ik}) - d_k$ whenever b_{ij} and b_{ik} are both nonzero.

Let $\{B_1, \dots, B_r\}$ denote a set of r independent column vectors of δ^t, $B_j = b_{j1}, \dots, b_{jn}$ for $1 \le j \le r$. Then, the determinant, Δ, of the matrix (b_{ij}) is a nonzero element of R. By Lemma 3.2, $0 \ne \overline{\Delta} \in R_\mu$ for almost every μ. Denoting $\overline{B}_i = (\overline{b}_{i1}, \dots, \overline{b}_{in})$ for $r < i \le n$, it follows that $S'_\mu = \{\overline{B}_{r+1}, \dots, \overline{B}_n\}$ forms a basis for $\ker(\overline{\delta}^t)$ in L_μ for almost every μ. If $A = (a_1, \dots, a_n)\in\ker(\overline{\delta}^t)$, there exists $0\ne b\in R_\mu$ such that bA lies in the R-module spanned by S'_μ. Thus S'_μ gives rise to a fundamental set of derivations corresponding to S_μ, for almost every μ. The derivation represented by B_i is just $D_i = \sum_{j=1}^n b_{ij}\partial/\partial X_j$. Since $\deg(b_{ij}) - \deg(X_j) = \deg(b_{ik}) - \deg(X_k)$ whenever b_{ij} and b_{ik} are both nonzero, D_i is homogeneous of degree n_i, for some constant n_i. It follows that $\overline{D}_i = \sum_{j=1}^n \overline{b_{ij}}\partial/\partial X_j$, the derivation represented by $\overline{B}_i$, is homogeneous of degree n_i for almost every μ. Thus, $N = \max_{r<j\le n}\{n_j\} = \max\{\deg(D_j) | D_j\in S_\mu\}$ can be computed in characteristic 0, independent of the choice of μ, for almost every μ. Let n be the product

of all prime numbers less than or equal to N. Then, in the dense open set $\operatorname{Spec}(A) - V(n)$, every maximal ideal μ has the property that the characteristic of R_μ is a prime $p > N$. Theorem 3.6 now follows from Theorem 2.10. □

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